Resolution strategy for the Hybridizable Discontinuous Galerkin system for solving Helmholtz elastic wave equations

Marie Bonnasse-Gahot, Henri Calandra, Julien Diaz, Stephane Lanteri

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M. Bonnasse-Gahot[1,2], H. Calandra[3], J. Diaz[1] and S. Lanteri[2]

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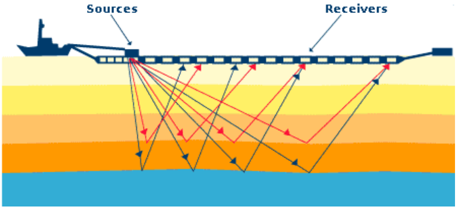
[3] *TOTAL Exploration-Production*

Depth Imaging Partnership

September 14, 2016

Motivations

Principles of seismic imaging



Motivations

Examples of seismic imaging campaigns

Motivations

Imaging methods

- Reverse Time Migration (RTM) : based on the **reversibility of wave equation**
- Full Wave Inversion (FWI) : **inversion process** requiring to solve many forward problems

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- **Time-domain** : imaging condition complicated but quite low computational cost
- **Harmonic-domain** : imaging condition simple but huge computational cost

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⚠ **Memory usage** ⚠

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Resolution of the forward problem of the inversion process

- Elastic wave propagation in the frequency domain : Helmholtz equation

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Resolution of the forward problem of the inversion process

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First order formulation of Helmholtz wave equations

$\mathbf{x} = (x, y, z) \in \Omega \subset \mathbb{R}^3$,

$$\begin{cases} i\omega\rho(\mathbf{x})\mathbf{v}(\mathbf{x}) = \nabla\cdot\underline{\underline{\sigma}}(\mathbf{x}) + f_s(\mathbf{x}) \\ i\omega\underline{\underline{\sigma}}(\mathbf{x}) = \underline{\underline{C}}(\mathbf{x})\ \underline{\underline{\varepsilon}}(\mathbf{v}(\mathbf{x})) \end{cases}$$

- $\mathbf{v}$: velocity vector
- $\underline{\underline{\sigma}}$: stress tensor
- $\underline{\underline{\varepsilon}}$: strain tensor

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Resolution of the forward problem of the inversion process

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- ρ : mass density
- $\underline{\underline{C}}$: elasticity tensor
- f_s : source term, $f_s \in L^2(\Omega)$

Approximation methods

Discontinuous Galerkin Methods

✓ unstructured tetrahedral meshes
✓ combination between FEM and finite volume method (FVM)
✓ *hp*-adaptivity
✓ easily parallelizable method

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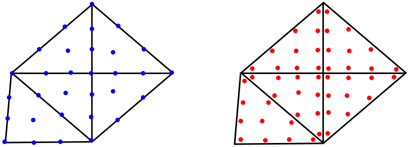

Approximation methods

Hybridizable Discontinuous Galerkin Methods

✓ same advantages as DG methods : unstructured tetrahedral meshes, *hp*-adaptivity, easily parallelizable method, discontinuous basis functions

✓ introduction of a new variable defined only on the interfaces

✓ lower number of coupled DOF than classical DG methods

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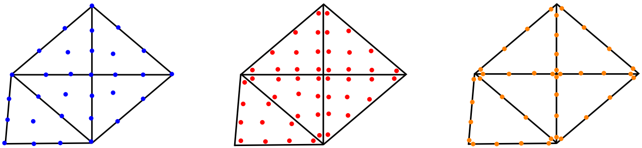

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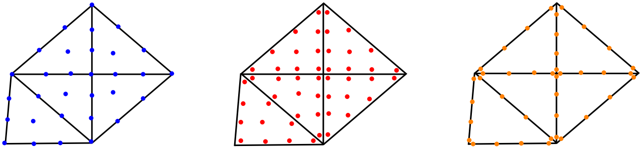
✓ lower number of coupled DOF than classical DG methods

✗ time-domain increases computational costs



Hybridizable Discontinuous Galerkin method

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- S. Lanteri, L. Li and R. Perrussel. Numerical investigation of a high order hybridizable discontinuous Galerkin method for 2d time-harmonic Maxwell's equations. *COMPEL,* 32(3)1112-1138, 2013.
- N.C. Nguyen, J. Peraire and B. Cockburn. High-order implicit hybridizable discontinuous Galerkin methods for acoustics and elastodynamics. *Journal of Computational Physics,* 230 :7151-7175, 2011
- N.C. Nguyen and B. Cockburn. Hybridizable discontinuous Galerkin methods for partial differential equations in continuum mechanics. *Journal of Computational Physics* 231 :5955–5988, 2012

Contents

HDG formulation of the equations

Local HDG formulation

$$\left\{ \begin{array}{rcl} i\omega\rho\mathbf{v} - \nabla \cdot \underline{\underline{\sigma}} & = & 0 \\ i\omega\underline{\underline{\sigma}} - \underline{\underline{C}}\,\underline{\underline{\varepsilon}}(\mathbf{v}) & = & 0 \end{array} \right.$$

HDG formulation of the equations

Local HDG formulation

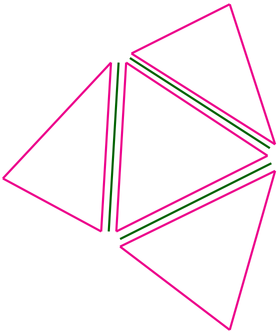
$$\left\{\begin{aligned} &\int_K i\omega\rho^K \mathbf{v}^K\cdot\mathbf{w} + \int_K \underline{\underline{\sigma}}^K : \nabla\mathbf{w} - \int_{\partial K} \widehat{\underline{\underline{\sigma}}}^{\partial K}\cdot\mathbf{n}\cdot\mathbf{w} = 0\\ &\int_K i\omega\underline{\underline{\sigma}}^K : \underline{\underline{\xi}} + \int_K \mathbf{v}^K\cdot\nabla\cdot\left(\underline{\underline{C}}^K\underline{\underline{\xi}}\right) - \int_{\partial K}\widehat{\mathbf{v}}^{\partial K}\cdot\underline{\underline{C}}^K\underline{\underline{\xi}}\cdot\mathbf{n} = 0\end{aligned}\right.$$

$\widehat{\underline{\underline{\sigma}}}^K$ and $\widehat{\mathbf{v}}^K$ are numerical traces of $\underline{\underline{\sigma}}^K$ and $\mathbf{v}^K$ respectively on ∂K

HDG formulation of the equations

We define :

$$\widehat{\mathbf{v}}^{\partial K} = \lambda^F, \quad \forall F \in \mathcal{F}_h,$$

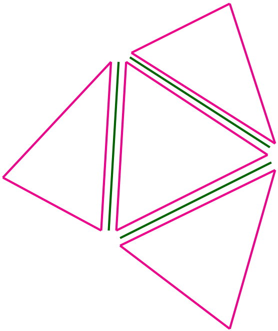


HDG formulation of the equations

We define :

$$\begin{aligned} \widehat{\mathbf{v}}^{\partial K} &= \lambda^F, & \forall F \in \mathcal{F}_h, \\ \widehat{\underline{\underline{\sigma}}}^{\partial K} \cdot \mathbf{n} &= \underline{\underline{\sigma}}^K \cdot \mathbf{n} - \tau \mathbf{I} \left(\mathbf{v}^K - \lambda^F \right), & \text{on } \partial K \end{aligned}$$

where τ is the stabilization parameter ($\tau > 0$)



HDG formulation of the equations

Local HDG formulation

$$\left\{\begin{array}{l}\displaystyle\int_K i\omega\rho^K \mathbf{v}^K\cdot\mathbf{w}-\int_K\left(\nabla\cdot\underline{\underline{\sigma}}^K\right)\cdot\mathbf{w}+\int_{\partial K}\tau\mathbf{I}\left(\mathbf{v}^K-\lambda^F\right)\cdot\mathbf{w}=0\\ \displaystyle\int_K i\omega\underline{\underline{\sigma}}^K:\underline{\underline{\xi}}+\int_K\mathbf{v}^K\cdot\nabla\cdot\left(\underline{\underline{C}}^K\underline{\underline{\xi}}\right)-\int_{\partial K}\lambda^F\cdot\underline{\underline{C}}^K\underline{\underline{\xi}}\cdot\mathbf{n}=0\end{array}\right.$$

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We define :

$$\underline{W}^K = \left(\underline{V_x}^K,\ \underline{V_y}^K,\ \underline{V_z}^K,\ \underline{\sigma_{xx}}^K,\ \underline{\sigma_{yy}}^K,\ \underline{\sigma_{zz}}^K,\ \underline{\sigma_{xy}}^K,\ \underline{\sigma_{xz}}^K,\ \underline{\sigma_{yz}}^K\right)^T$$

$$\underline{\Lambda} = \left(\underline{\Lambda}^{F_1},\ \underline{\Lambda}^{F_2}, \ldots,\ \underline{\Lambda}^{F_{n_f}}\right)^T, \text{ where } n_f = \mathrm{card}(\mathcal{F}_h)$$

Discretization of the local HDG formulation

$$\mathbb{A}^K \underline{W}^K + \sum_{F \in \partial K} \mathbb{C}^{K,F} \underline{\Lambda} = 0$$

HDG formulation of the equations

Local HDG formulation

$$\begin{cases} \displaystyle\int_K i\omega\rho^K \mathbf{v}^K \cdot \mathbf{w} - \int_K \left(\nabla \cdot \underline{\underline{\sigma}}^K\right) \cdot \mathbf{w} + \int_{\partial K} \tau \mathbf{I}\left(\mathbf{v}^K - \lambda^F\right) \cdot \mathbf{w} = 0 \\ \displaystyle\int_K i\omega\underline{\underline{\sigma}}^K : \underline{\underline{\xi}} + \int_K \mathbf{v}^K \cdot \nabla \cdot \left(\underline{\underline{C}}^K \underline{\underline{\xi}}\right) - \int_{\partial K} \lambda^F \cdot \underline{\underline{C}}^K \underline{\underline{\xi}} \cdot \mathbf{n} = 0 \end{cases}$$

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Discretization of the local HDG formulation

$$\mathbb{A}^K \underline{W}^K + \mathbb{C}^K \underline{\Lambda} = 0$$

HDG formulation of the equations

Transmission condition

In order to determine λ^F, the continuity of the normal component of $\widehat{\underline{\underline{\sigma}}}^{\partial K}$ is weakly enforced, rendering this numerical trace conservative :

$$\int_F [\![\widehat{\underline{\underline{\sigma}}}^{\partial K} \cdot \mathbf{n}]\!] \cdot \eta = 0$$

HDG formulation of the equations

Transmission condition

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$$\int_F [\![\underline{\underline{\widehat{\sigma}}}^{\partial K} \cdot \mathbf{n}]\!] \cdot \eta = 0$$

Discretization of the transmission condition

$$\sum_{K \in \mathcal{T}_h} \left[\mathbb{B}^K \underline{W}^K + \mathbb{L}^K \underline{\Lambda} \right] = 0$$

HDG formulation of the equations

Global HDG discretization

$$\left\{ \begin{array}{l} \mathbb{A}^K \underline{W}^K + \mathbb{C}^K \underline{\Lambda} = 0 \\ \displaystyle\sum_{K \in \mathcal{T}_h} \left[\mathbb{B}^K \underline{W}^K + \mathbb{L}^K \underline{\Lambda} \right] = 0 \end{array} \right.$$

HDG formulation of the equations

Global HDG discretization

$$\begin{cases} \underline{W}^K = -(\mathbb{A}^K)^{-1}\mathbb{C}^K\underline{\Lambda} \\ \displaystyle\sum_{K\in\mathcal{T}_h} \left[\mathbb{B}^K\underline{W}^K + \mathbb{L}^K\underline{\Lambda}\right] = 0 \end{cases}$$

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Global HDG discretization

$$\sum_{K \in \mathcal{T}_h} \left[-\mathbb{B}^K (\mathbb{A}^K)^{-1} \mathbb{C}^K + \mathbb{L}^K \right] \underline{\Lambda} = 0$$

Main steps of the HDG algorithm

1. Construction of the global matrix $\mathbb{M}$

with $\mathbb{M} = \sum_{K \in \mathcal{T}_h} \left[-\mathbb{B}^K (\mathbb{A}^K)^{-1} \mathbb{C}^K + \mathbb{L}^K \right]$

for $K = 1$ to Nb_{tri} **do**

Computation of matrices $\mathbb{B}^K, (\mathbb{A}^K)^{-1}, \mathbb{C}^K$ and $\mathbb{L}^K$

Construction of the corresponding section of $\mathbb{M}$

end for

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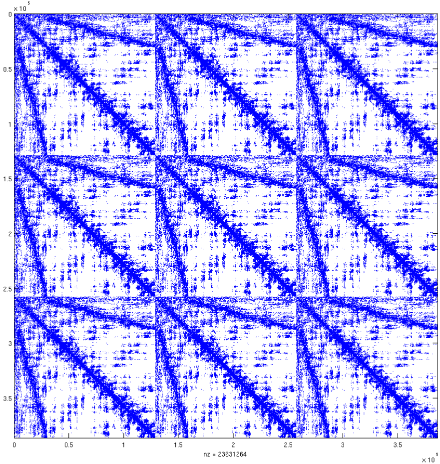
for $K = 1$ to Nb_{tri} **do**
  Compute $\underline{W}^K = -(\mathbb{A}^K)^{-1}\mathbb{C}^K\underline{\Lambda}$
end for

MaPhys Vs MUMPS

Pattern of the HDG global matrix for $\mathbb{P}_1$ interpolation and for a 3D mesh composed of 21 000 elements

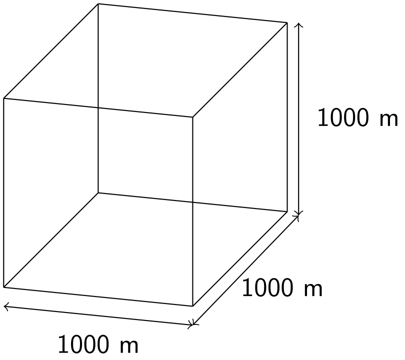


MaPhys Vs MUMPS

Software packages for solving systems of linear equations $Ax = b$, where A is a sparse matrix

- MUMPS (MUltifrontal Massively Parallel sparse direct Solver) :
  - Direct factorization $A = LU$ or $A = LDL^T$
  - Multifrontal approach
- MaPhys (Massively Parallel Hybrid Solver) :
  - Direct and iterative methods
  - non-overlapping algebraic domain decomposition method (Schur complement method)
  - resolution of each local problem thanks to direct solver such as MUMPS or PaStiX.

3D plane wave in an homogeneous medium



Configuration of the computational domain Ω .

- Physical parameters :
  - $\rho = 1$ kg.m^{-3}
  - $\lambda = 16$ GPa
  - $\mu = 8$ GPa
- Plane wave :

$$u = \nabla e^{i(k_x x + k_y y + k_z z)}$$

where $k_x = \dfrac{\omega}{v_p} \cos\theta_0 \cos\theta_1$,

$k_y = \dfrac{\omega}{v_p} \sin\theta_0 \cos\theta_1$, and

$k_z = \dfrac{\omega}{v_p} \sin\theta_1$

- $\omega = 2\pi f, f = 8$ Hz
- $\theta_0 = 30^\circ, \theta_1 = 0^\circ$
- Mesh composed of 21 000 elements

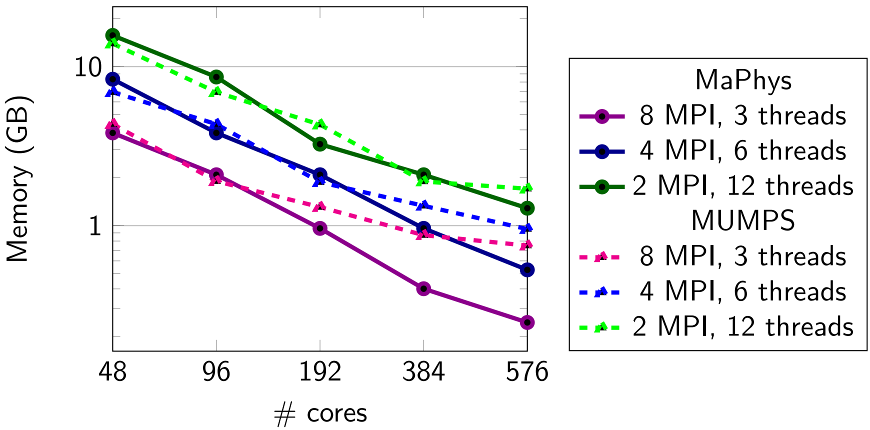
Cluster configuration

Features of the nodes :

- 2 Dodeca-core Haswell Intel Xeon E5-2680
- Frequency : 2,5 GHz
- RAM : 128 Go
- Storage : 500 Go
- Infiniband QDR TrueScale : 40Gb/s
- Ethernet : 1Gb/s

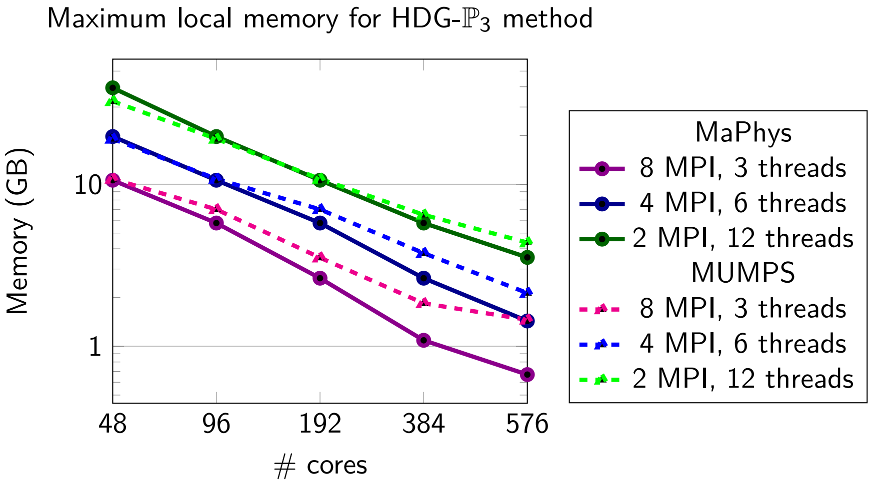
3D Plane wave : Memory consumption

Maximum local memory for HDG-$\mathbb{P}_2$ method



(matrix order = 772 416, # nz=107 495 424)

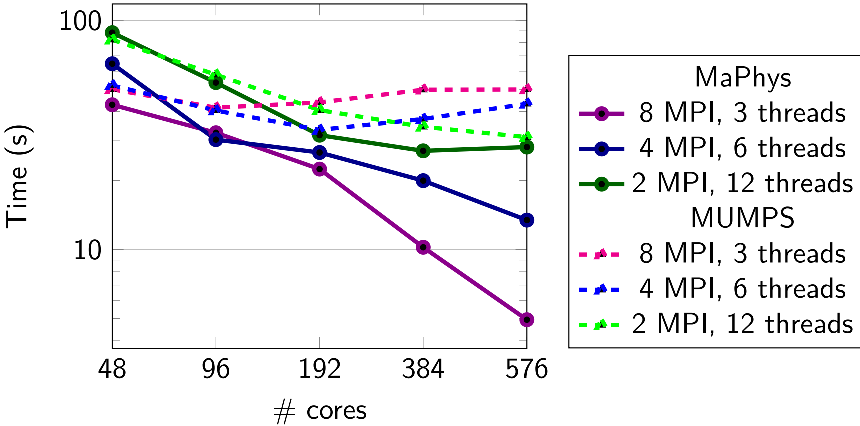
3D Plane wave : Memory consumption



(matrix order = 1 287 360, # nz=298 598 400)

3D Plane wave : Execution time

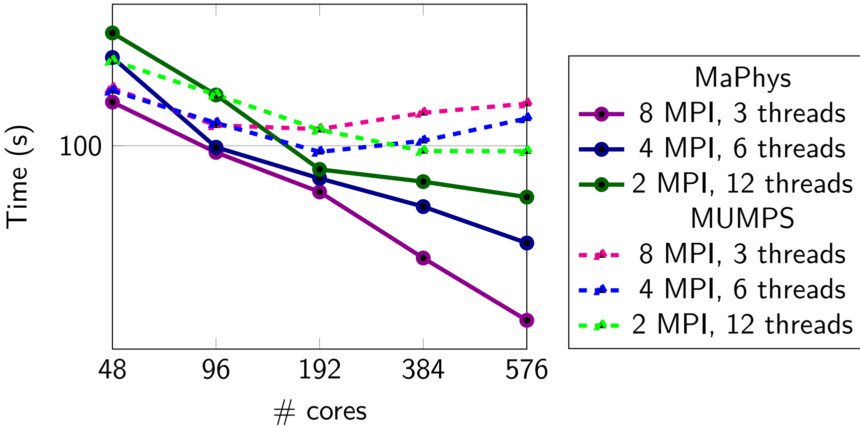
Execution time for the resolution of the HDG-$\mathbb{P}_2$ system



(matrix order = 772 416, # nz=107 495 424)

3D Plane wave : Execution time

Execution time for the resolution of the HDG-$\mathbb{P}_3$ system



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Conclusion-Perspectives

- HDG method implemented in Total program (WP6)
- more detailled analysis of the comparison between MUMPS and MaPhys (WP3)
- comparison between to PaStiX solver
- extension to elasto-acoustic case
- call for projects PRACE to test bigger test-cases

Thank you !

Factorization time (s) for the HDG-$\mathbb{P}_2$ system
(Matrix order = 772 416, # nz = 107 495 424)

	2 nodes		4 nodes		8 nodes		16
	Maphys	Mumps	Maphys	Mumps	Maphys	Mumps	Maphys
8 MPI/n., 3 t./MPI	21.77	42.55	7.18	35.06	2.62	37.54	1.32
4 MPI/n. 6 t./MPI	42.37	44.66	14.05	33.69	5.28	26.80	2.48
2 MPI/n. 12 t./MPI	70.20	69.48	29.11	49.69	10.79	33.44	4.22